York University
EECS 2011Z Winter 2015 – Problem Set 1
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1. Prove whether each of the following is true or false. x and y are real variables.

   1) $\forall x\ \exists y\ x \cdot y = 5$
   2) $\exists y\ \forall x\ x \cdot y = 5$
   3) $\forall x\ \exists y\ x \cdot y = 0$
   4) $\exists y\ \forall x\ x \cdot y = 0$
   5) $\exists a\ \forall x\ \exists y\ [x = a \text{ or } x \cdot y = 5]$

   - Answer:
     (a) $\forall x\ \exists y\ x \cdot y = 5$ is false. Let $x = 0$. Then y must be $\frac{5}{0}$, which is impossible.
     (b) $\exists y\ \forall x\ x \cdot y = 5$ is false. Let y be an arbitrary real value and let $x = \frac{6}{y}$ if $y \neq 0$ and $x = 0$ if $y = 0$. Then $x \cdot y \neq 5$.
     (c) $\forall x\ \exists y\ x \cdot y = 0$ is true. Let x be an arbitrary real value and let $y = 0$. Then $x \cdot y = 0$.
     (d) $\exists y\ \forall x\ x \cdot y = 0$ is true. Let $y = 0$ and let x be an arbitrary real value. Then $x \cdot y = 0$.
     (e) $\exists a\ \forall x\ \exists y\ [x = a \text{ or } x \cdot y = 5]$ is true. Let $a = 0$. Let x be an arbitrary real value. If $x = 0$ then $[x = 0 \text{ or } x \cdot y = 5]$ is true because of the left. If $x \neq 0$ then let $y = \frac{5}{x}$ and $[x = 0 \text{ or } x \cdot y = 5]$ is true because of the right.

2. Asymptotic Running Times

   True or False? All logarithms are base 2. No justification is necessary.

   (a) $5n^2 \log n \in O(n^2)$
   - Answer: False. It is a factor of $\log n$ too big.

   (b) $4^{8n} \in O(8^{4n})$
   - Answer: False: $4^{8n} = 2^{16n}$, but $8^{4n} = 2^{12n}$.

   (c) $2^{10 \log n} + 100(\log n)^{11} \in O(n^{10})$
   - Answer: True: $2^{10 \log n} = n^{10}$, $100(\log n)^{11} \in O(n^{10})$.

   (d) $2n^2 \log n + 3n^2 \in \Theta(n^3)$
   - Answer: False: $2n^2 \log n + 3n^2 \in O(n^3)$, but $2n^2 \log n + 3n^2 \notin \Omega(n^3)$.

3. Big-Oh Definition

   Fill in the blanks:

   $f(n) \in O(g(n))$ iff ____ $c > 0$, ____$n_0 > 0$, such that ____ n ____$n_0, f(n)$____ $cg(n)$

   - Answer:
     $f(n) \in O(g(n))$ iff $\underline{\exists}$ $c > 0$, $\underline{\exists}$ $n_0 > 0$, such that $\underline{\forall}$ n $\underline{\geq}$ n_0, $f(n)$ $\underline{\leq}$ $cg(n)$

4. Order the following functions by increasing asymptotic growth rate:

$$\begin{array}{ccc} 4n\log n + 2n & 2^{10} & 2^{\log n} \\ 3n + 100\log n & 4n & 2^n \\ n^2 + 10n & n^3 & n\log n \end{array}$$

   - Answer:
     (a) 2^{10}
     (b) $2^{\log n}$
     (c) $3n + 100\log n$
     (d) $4n$
     (e) $n\log n$
     (f) $4n\log n + 2n$
     (g) $n^2 + 10n$
     (h) n^3
     (i) 2^n

5. Prove that $n\log n - n$ is $\Omega(n)$.

   - Answer:
     $\log n \geq 2\ \forall n \geq 4$. Thus $n\log n - n \geq n\ \forall n \geq 4 \to n\log n - n \in \Omega(n)$.

6. Prove that if $d(n)$ is $\mathsf{O}(f(n))$ and $e(n)$ is $\mathsf{O}(g(n))$, then the product $d(n)e(n)$ is $\mathsf{O}(f(n)g(n))$.

   - Answer:
     $d(n) \in \mathcal{O}\left(f(n)\right) \to \exists c_1, n_1 > 0 : d(n) \leq c_1 f(n) \forall n \geq n_1$.
     Similarly,
     $e(n) \in \mathcal{O}\left(g(n)\right) \to \exists c_2, n_2 > 0 : e(n) \leq c_2 f(n) \forall n \geq n_2$.
     Thus, letting $c_0 = c_1c_2$ and $n_0 = \max\{n_1, n_2\}$, we have
     $d(n)e(n) \leq c_0 f(n)g(n) \forall n \geq n_0 \to d(n)e(n) \in \mathcal{O}\left(f(n)g(n)\right)$.

7. An evil king has n bottles of wine, and a spy has just poisoned one of them. Unfortunately, they dont know which one it is. The poison is very deadly; just one drop diluted even a billion to one will still kill. Even so, it takes a full month for the poison to take effect. Design a scheme for determining exactly which one of the wine bottles was poisoned in just one months time while expending only $\mathsf{O}(\log n)$ royal tasters. State your scheme briefly, in English.

   - Answer: Label each bottle from 0 to $n - 1$, and consider each as a binary number consisting of $\lceil \log n \rceil$ bits. Now assemble $\lceil \log n \rceil$ royal goblets. Take a drop from each of the bottles whose lowest order bit is set and deposit in the first goblet. Then take a drop from each bottle whose 2nd bit is set and deposit in the second goblet. Continue in similar fashion through the highest-order bit. Now hand each of the royal tasters one of the goblets and command them to drink. Note that there is now a 1:1 correspondence between bits and tasters. In a month, some of your tasters will drop dead. Set the corresponding bits to 1, and all other bits to 0. The resulting binary number identifies the poisoned bottle. Long live the king!

8. Asymptotic Running Times

True or False? All logarithms are base 2. No justification is necessary.

(a) $2^n \in \Omega\left(n^3\right)$

- Answer: True

(b) $3n^3 + 17n^2 \in O\left(n^3\right)$

- Answer: True. For example, $3n^3 + 17n^2 \leq 20n^3 \forall n \geq 1$.

(c) $5n^2 \log n \in O(n^2)$

- Answer: False. It is a factor of $\log n$ too big.

(d) $2^{10 \log n} + 100(\log n)^{11} \in O(n^{10})$

- Answer: True: $2^{10 \log n} = n^{10}$, $100(\log n)^{11} \in O(n^{10})$.

(e) $2n^2 \log n + 3n^2 \in \Theta(n^3)$

- Answer: False: $2n^2 \log n + 3n^2 \in O(n^3)$, but $2n^2 \log n + 3n^2 \notin \Omega(n^3)$.

9. Show that n^2 is $\Omega\left(n \log n\right)$.

- Answer: We seek a $c > 0, n_0 > 0 : \forall n \geq n_0,\ n^2 \geq cn \log n \leftrightarrow n \geq c \log n$. Let $c = 1$. Then we require that $n \geq \log n$. This is satisfied $\forall n \geq 1$. Thus n^2 is $\Omega\left(n \log n\right)$.